Readers' Forum

Brief discussion of previous investigations in the aerospace sciences and technical comments on papers published in the AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

Comment on "Penalized Weighted Residual Method for the Initial Value Problem"

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IN a recent paper dealing with the use of the Hamiltonian law of varying action for the calculation of dynamic systems, Kim and Cho[1] propose to introduce the initial values of the problem by using a penalty function. The justification for the application of such a function is given in the paper[1] as follows: "Because the initial displacement $u(t_0)$ is specified, the variation of the initial displacement $\delta u(t_0)$ must be zero for kinematically admissible variations. Therefore the initial velocity cannot be considered appropriately in $m\dot{u}\delta u|_{t_0} = 0$. Thus the theories using Hamilton's law of varying action must consider tentatively $\delta u(t_0)$ as free variation independent of $u(t_0)$ and impose $u(t_0)$, $\dot{u}(t_0)$ at the approximate stage."

There is a basic misconception in the paper.[1] The idea[2–4] of using a variation δu independent of u has a deeper importance than the one assumed in the paper.[1] The independent δu is applied to stabilize the calculation process.[2–4] For example, in Ref. 3 the second derivative of u was used as δu. In this way the calculation process has been stabilized (it is interesting that Gauss[5] in his principle treated $\ddot{u}$ as an independent variable; see also Ref. 6). In Ref. 3 a method for unconditional stability was also proposed. An analysis of the stability of the calculation process was performed in Refs. 3 and 4. Such an instability analysis does not appear in the paper of Kim and Cho.[1]

References

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Reply by the Authors to M. Baruch

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Introduction

IN his Comment on our paper,[1] Baruch argues about the interpretation of using variation δu independent of u in the paper[2] and the stability of our method. We will answer these questions in the following two sections.

Interpretation of Variation δu

The statement of the paper[1] quoted by the Comment deals with a natural imposition of the initial velocity in the variational form based on displacement regardless of approximation, retaining the physical meaning of kinematically admissible variation $\delta u(t_0)$. When u and $\dot{u}$ denoted displacement and velocity and m, c, k, and f were the mass, damping, stiffness, and external force, respectively, the variational statement in the time interval between t_0 and t_f was written as

$$\int_{t_0}^{t_f} (-m\dot{u}\delta\dot{u} + c\dot{u}\delta u + ku\delta u - f\delta u)\,\mathrm{d}t + m\dot{u}\delta u\big|_{t_f} - m\dot{u}\delta u\big|_{t_0} = 0 \tag{1}$$

in the paper by Riff and Baruch.[2] Two terms at t_0 and t_f in the variational statement (1) are the feature that makes it different from the conventional variational principle with the specification of initial conditions at t_0. Keeping the terms means that the variation $\delta u(t_0)$ is not zero even though $u(0)$ is specified with fixed value. It was explained in the Comment that the variation $\delta u(t)$ independent of $u(t)$ is used only for stability of numerical scheme. But it is difficult to agree with the preceding explanation. It is better to not mention the numerical stability in the early formulation stage because the stability is largely dependent on the specific numerical schemes used after the formulation. In fact, we were able to obtain a stable procedure, as explained in the next section, without the inclusion of $m\dot{u}\delta u$ at t_0, following the strict Galerkin approximation procedure.

In the paper,[1] we were concerned with the displacement-based variational statement (1) anticipating the use of the Galerkin approximation. Our comment to the variational statement (1) in the paper[1] was made by the same line of thought. In the penalized weighted residual formulation,[1] the meaning of the kinematically admissible variation is consistently used and the initial velocity is imposed by penalized form to satisfy the initial velocity in the sense of average. From the formulation, both the equation of motion and the initial condition can be reconstructed to their original states. The

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restoration of the equation of motion and the initial condition from the penalized weighted residual formulation shows the appropriate imposition of the initial condition in the early formulation stage. Moreover, the penalized weighted residual formulation can be easily approximated by the C^0 approximating functions, which satisfy the smoothness requirement of weak formulation and allow weak solution like discontinuity of velocities frequently encountered in dynamic problems due to impulse loading.

Stability of Penalized Weighted Residual Method

Because the observation of stability characteristics was not the objective of the paper,[1] the stability analysis was not presented in the paper.[1] However, the result of recent research[3] may be an answer for the question about the stability of the penalized weighted residual method. In recent research, the stability analysis is carried out by way of a newly developed alternative step-by-step computing algorithm of the penalized weighted residual method. The resulting equation of penalized weighted residual method discretized by the αth-order C^0 element is as follows:

$$\left[\int_{t_0}^{t_f}(-\dot{\phi}_i\dot{\phi}_j\boldsymbol{M}+\phi_i\dot{\phi}_j\boldsymbol{C}+\phi_i\phi_j\boldsymbol{K})\,\mathrm{d}t\right]\boldsymbol{u}_j + \left[\phi_i(t_f)\dot{\phi}_j(t_f)\boldsymbol{M}+\frac{1}{\varepsilon}\phi_i(t_f)\dot{\phi}_j(t_0)\boldsymbol{I}\right]\boldsymbol{u}_j = \left(\int_{t_0}^{t_f}\phi_i\,\boldsymbol{f}\,\mathrm{d}t\right) + \left[\frac{1}{\varepsilon}\phi_i(t_f)\boldsymbol{I}\boldsymbol{v}_0 - \frac{1}{\varepsilon}\phi_i(t_f)\dot{\phi}_0(t_0)\boldsymbol{I}\boldsymbol{u}_0 - \phi_i(t_f)\dot{\phi}_0(t_f)\boldsymbol{M}\boldsymbol{u}_0\right] - \left[\int_{t_0}^{t_f}(-\dot{\phi}_i\dot{\phi}_0\boldsymbol{M}+\phi_i\dot{\phi}_0\boldsymbol{C}+\phi_i\phi_0\boldsymbol{K})\,\mathrm{d}t\right]\boldsymbol{u}_0 \tag{2}$$

where $1 \le i, j \le N\alpha$ (where N is the number of time elements) and $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$, and $\boldsymbol{I}$ denote mass, damping, stiffness, and identity matrices, respectively; $\boldsymbol{v}_0$ and $\boldsymbol{f}$ are the initial velocity nodal vector and the external force vector; $\boldsymbol{u}_i$ denotes the displacement nodal vector at time i; and ϕ and ε are the interpolating function and the penalty parameter, respectively.

The alternative computing algorithm is derived from the observation of the sparse nature of the resulting matrix and the fact that the penalized terms are much larger than the other terms in the resulting algebraic equation. For the αth-order C^0 time element, it is described as follows:

1) Find $\boldsymbol{U}_1$ with the given initial condition $\boldsymbol{U}_0$ such that $\Xi_1^{(s)}\boldsymbol{U}_1 + \Xi_0^{(s)}\boldsymbol{U}_0 = \boldsymbol{F}_1$.

2) For $1 \le k \le (N-1)$, find the next step solution $\boldsymbol{U}_{k+1}$ with the obtained $\boldsymbol{U}_k$ and $\boldsymbol{U}_{k-1}$ such that

$$\Xi_{k+1}^{(f)}\boldsymbol{U}_{k+1} + \Xi_k^{(c)}\boldsymbol{U}_k + \Xi_{k-1}^{(p)}\boldsymbol{U}_{k-1} = \boldsymbol{F}_{k+1}$$

where

$$\Xi_1^{(s)} = \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \cdots & \boldsymbol{p}_\alpha \\ \boldsymbol{a}_{1,1} & \boldsymbol{a}_{1,2} & \cdots & \boldsymbol{a}_{1,\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{a}_{\alpha-1,1} & \boldsymbol{a}_{\alpha-1,2} & \cdots & \boldsymbol{a}_{\alpha-1,\alpha} \end{bmatrix}$$

$$\Xi_0^{(s)} = \begin{bmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{p}_0 \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{a}_{1,0} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{a}_{\alpha-1,0} \end{bmatrix}$$

$$\Xi_{k+1}^{(f)} = \begin{bmatrix} \boldsymbol{a}_{k\alpha,k\alpha+1} & \boldsymbol{a}_{k\alpha,k\alpha+2} & \cdots & \boldsymbol{a}_{k\alpha,(k+1)\alpha} \\ \boldsymbol{a}_{k\alpha+1,k\alpha+1} & \boldsymbol{a}_{k\alpha+1,k\alpha+2} & \cdots & \boldsymbol{a}_{k\alpha+1,(k+1)\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{a}_{(k+1)\alpha-1,k\alpha+1} & \boldsymbol{a}_{(k+1)\alpha-1,k\alpha+2} & \cdots & \boldsymbol{a}_{(k+1)\alpha-1,(k+1)\alpha} \end{bmatrix}$$

$$\Xi_k^{(c)} = \begin{bmatrix} \boldsymbol{a}_{k\alpha,(k-1)\alpha+1} & \cdots & \boldsymbol{a}_{k\alpha,k\alpha-1} & \boldsymbol{a}_{k\alpha,k\alpha} \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{a}_{k\alpha+1,k\alpha} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{a}_{(k+1)\alpha-1,k\alpha} \end{bmatrix}$$

$$\Xi_{k-1}^{(p)} = \begin{bmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{a}_{k\alpha,(k-1)\alpha} \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

and

$$\boldsymbol{a}_{ij} = \int_{t_0}^{t_f}[-\dot{\phi}_i\dot{\phi}_j\boldsymbol{M}+\phi_i\dot{\phi}_j\boldsymbol{C}+\phi_i\phi_j\boldsymbol{K}]\,\mathrm{d}t$$

$$\boldsymbol{p}_j = (1/\varepsilon)\dot{\phi}_j(t_0)\boldsymbol{I} \qquad (0 \le j \le \alpha \text{ and } \boldsymbol{I} \text{ is the identity matrix})$$

$$\boldsymbol{F}_1 = \left(\frac{1}{\varepsilon}\boldsymbol{v}_0^T, \int_{t_0}^{t_f}\phi_1\boldsymbol{f}^T\,\mathrm{d}t, \int_{t_0}^{t_f}\phi_2\boldsymbol{f}^T\,\mathrm{d}t, \ldots, \int_{t_0}^{t_f}\phi_{\alpha-1}\boldsymbol{f}^T\,\mathrm{d}t\right)^T$$

$$\boldsymbol{F}_{k+1} = \left(\int_{t_0}^{t_f}\phi_{k\alpha}\boldsymbol{f}^T\,\mathrm{d}t, \int_{t_0}^{t_f}\phi_{k\alpha+1}\boldsymbol{f}^T\,\mathrm{d}t, \ldots, \int_{t_0}^{t_f}\phi_{(k+1)\alpha-1}\boldsymbol{f}^T\,\mathrm{d}t\right)^T \qquad (1 \le k \le N-1)$$

$$\boldsymbol{U}_0 = (\boldsymbol{0}^T, \boldsymbol{0}^T, \ldots, \boldsymbol{0}^T, \boldsymbol{u}_0^T)^T \qquad \boldsymbol{U}_1 = (\boldsymbol{u}_1^T, \boldsymbol{u}_2^T, \ldots, \boldsymbol{u}_\alpha^T)^T$$

$$\boldsymbol{U}_{k+1} = (\boldsymbol{u}_{k\alpha+1}^T, \boldsymbol{u}_{k\alpha+2}^T, \ldots, \boldsymbol{u}_{(k+1)\alpha}^T)^T \qquad (1 \le k \le N-1)$$

Using the algorithm, the stability characteristics can be observed. If an integration method is unconditionally stable for a system of one degree of freedom, it guarantees the unconditional stability of the numerical solution of a system having multiple degrees of freedom. Thus the stability analysis of one degree of freedom is sufficient for the proof. Let us consider the free oscillating system

$$\ddot{u} + \omega^2 u = 0 \tag{3}$$

where ω denotes the frequency of the system and $\ddot{u}$ and u denote acceleration and displacement. By applying the algorithm to the equation, the same recurrence formula is obtained at each step except the initial step,

$$\Xi_{k+1}^{(f)}\boldsymbol{U}_{k+1} + \Xi_k^{(c)}\boldsymbol{U}_k + \Xi_{k-1}^{(p)}\boldsymbol{U}_{k-1} = \boldsymbol{0} \tag{4}$$

The amplification of $\boldsymbol{U}$ determines the stability of the numerical solution. If the amplification is greater than 1, it means that the numerical solution is unstable. For the analysis, the amplification factor λ is introduced and the following is assumed[4]:

$$\boldsymbol{U}_{k-1} = \boldsymbol{e}, \qquad \boldsymbol{U}_k = \lambda\boldsymbol{e}, \qquad \boldsymbol{U}_{k+1} = \lambda^2\boldsymbol{e} \tag{5}$$

Then the equation for obtaining the amplication factor λ is reduced to

$$\det\left[\lambda^2\Xi_{k+1}^{(f)} + \lambda\Xi_k^{(c)} + \Xi_{k-1}^{(p)}\right] = 0 \tag{6}$$

For the first-order C^0 approximation, the results are obtained as follows:

Case of full Gauss integration in the time domain:

$$\lambda_{1,2} = \frac{6 - 2\omega^2\Delta t^2 \pm \sqrt{36\omega^2\Delta t^2 + 3\omega^4\Delta t^4}}{6 + \omega^2\Delta t^2} \tag{7}$$

Case of reduced Gauss integration in the time domain:

$$\lambda_{1,2} = \frac{4 - \omega^2\Delta t^2 \pm i4\omega\Delta t}{4 + \omega^2\Delta t^2} \tag{8}$$

where Δt denotes the size of the time element. For the case of full Gauss integration, the solution is stable if Δt satisfies $-36\omega^2\Delta t^2 + 3\omega^4\Delta t^4 \leq 0$. Thus the case is conditionally stable with the critical time element size $\Delta t_{ct} = \sqrt{(3)}T/\pi = 2\sqrt{(3)}/\omega$ (T is the period). The case of reduced Gauss integration gives the amplification factor of magnitude 1. Thus an unconditional stability is obtained, which gives a stable solution regardless of time element size. If the second-order element is used with reduced integration, the amplification factor is reduced to

$$\lambda_{1,2} = \frac{[9 - 15\omega^2(\Delta t/2)^2 + \omega^4(\Delta t/2)^4] \pm i\{3\omega\Delta t[\omega^2(\Delta t/2)^2 - 3]\}}{[9 + 3\omega^2(\Delta t/2)^2 + \omega^4(\Delta t/2)^4]}$$

$$\lambda_{3,4} = 0 \tag{9}$$

If the initial displacement and velocity are zero, it corresponds to the eigenspace of λ_3 and λ_4, and the system does not move. (The corresponding amplification is zero.) If the initial condition is not zero, the magnitude of amplification factor is 1, and a stable solution is guaranteed. Likewise, for higher-order C^0 elements, the full Gauss integration in the time domain with the penalized weighted residual formulation results in a conditionally stable method having critical time element size, and the reduced Gauss integration in the time domain gives unconditional stability.

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